```
\begin{definition}
  \label{def:boundedseq}
  We denote by $\BoundedSeqs$ the set which satisfies
  \begin{equation}
    \BoundedSeqs
    =
    \{(a=(a_n)_{n\in\N}\colon\N\to\R)
      \colon \sup\{\abs{a_n}\colon n\in\N\} < \infty\}.
  \end{equation}
\end{definition}
```

Definition 1. We denote by ℓ^∞ the set which satisfies

$$\ell^\infty = \{(a = (a_n)_{n\in\mathbb{N}}\colon \mathbb{N} \to \mathbb{R})\colon \sup\{|a_n|\colon n \in \mathbb{N}\} < \infty\}. \tag{1}$$

```
\begin{athm}{lemma}{lem:monoconv}
  Let $a\in\BoundedSeqs$
  and assume that
    for all
      $n\in\N$
    it holds that
      $a_{n+1}\geq a_n$ \cfload.
  Then
    $a$ converges
  and
  \begin{equation}
    \lim_{n\to\infty}a_n = \sup(\{a_n\colon n\in\N\})
    \ifnocf.
  \end{equation}
  \cfout[.]
\end{athm}
```

Lemma 2. *Let $a \in \ell^\infty$ and assume that for all $n \in \mathbb{N}$ it holds that $a_{n+1} \geq a_n$ (cf. Definition 1). Then a converges and*

$$\lim_{n\to\infty} a_n = \sup(\{a_n\colon n \in \mathbb{N}\}). \tag{2}$$

```
\begin{aproof}
  Throughout this proof let
  \begin{equation}
    \llabel{sup}
    L = \sup(\{a_n\colon n\in\N\})\dott
  \end{equation}
```

Proof of Lemma 2. Throughout this proof let

$$L = \sup(\{a_n\colon n \in \mathbb{N}\}). \tag{3}$$

```
\argument{
  \lref{sup};
} {
  that for all $n\in\N$ it holds that
  \begin{equation}
    \llabel{lim1}
    a_n\leq L\dott
  \end{equation}
}
```

Observe that (3) ensures that for all $n \in \mathbb{N}$ it holds that

$$a_n \leq L. \tag{4}$$

```
\argument{
  \lref{sup};
  the assumption that $a\in\BoundedSeqs$;
} {
  that
  \llabel{Lfin}
  $
    L<\infty
  $\dott
}
```

Next, note that (3) and the assumption that $a \in \ell^\infty$ imply that $L < \infty$.

```
\argument{
  \lref{sup};
  \lref{Lfin};
} {
  that for all
    $\eps\in(0,\infty)$
  there exists
    $m\in\N$
  such that
  \begin{equation}
    \llabel{a1}
    a_m>L-\eps\dott
  \end{equation}
}
```

Combining this and (3) shows that for all $\varepsilon \in (0, \infty)$ there exists $m \in \mathbb{N}$ such that

$$a_m > L - \varepsilon. \tag{5}$$

```
\argument{
  the assumption that
    for all
      $n\in\N$
    it holds that
      $a_{n+1}\geq a_n$;
  induction
} {
  that for all
    $m\in\N$,
    $n\in\N\cap[m,\infty)$
  it holds that
  \begin{equation}
    \llabel{a3}
    a_n\geq a_m\dott
  \end{equation}
}
```

In the next step we combine the assumption that for all $n \in \mathbb{N}$ it holds that $a_{n+1} \geq a_n$ and induction to obtain that for all $m \in \mathbb{N}$, $n \in \mathbb{N} \cap [m, \infty)$ it holds that

$$a_n \geq a_m. \tag{6}$$

```
\argument{
  \lref{a1};
  \lref{a3}
} {
  that for all
    $\eps\in(0,\infty)$
  there exists
    $m\in\N$
  such that for all
    $n\in\N\cap[m,\infty)$
  it holds that
  \llabel{a2}
  $a_n>L-\eps$\dott
}
```

This and (5) prove that for all $\varepsilon \in (0, \infty)$ there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N} \cap [m, \infty)$ it holds that $a_n > L - \varepsilon$.

```
\argument{
  \lref{a2};
  \lref{lim1}
} {
  that for all
    $\eps\in(0,\infty)$
  there exists
    $m\in\N$
  such that for all
    $n\in\N\cap[m,\infty)$
  it holds that
  \begin{equation}
    \llabel{lim}
    \abs{a_n-L}<\eps\dott
  \end{equation}
}
```

This and (4) demonstrate that for all $\varepsilon \in (0, \infty)$ there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N} \cap [m, \infty)$ it holds that

$$|a_n - L| < \varepsilon. \tag{7}$$

```
\argument{
  \lref{lim}
} {
  that $a$ converges and $\lim_{n\to\infty}a_n = L$\dott
}
\end{aproof}
```

Hence, we obtain that a converges and $\lim_{n\to\infty} a_n = L$. The proof of Lemma 2 is thus complete. ☐

```
\begin{athm}{theorem}{thm:bolzanoweierstrass}
  [Bolzano--Weierstrass theorem]
  Let $a\in\BoundedSeqs$ \cfload.
  Then there exists
    $n=(n_k)_{k\in\N}\colon\N\to\N$
  such that it holds for all
    $k\in\N$
  that
    $n_{k+1}>n_k$
  and
    $(a_{n_l})_{l\in\N}$ converges.
\end{athm}
```

Theorem 3 (Bolzano–Weierstrass theorem)**.** *Let $a \in \ell^\infty$ (cf. Definition 1). Then there exists $n = (n_k)_{k\in\mathbb{N}} \colon \mathbb{N} \to \mathbb{N}$ such that it holds for all $k \in \mathbb{N}$ that $n_{k+1} > n_k$ and $(a_{n_l})_{l\in\mathbb{N}}$ converges.*

```
\begin{aproof}
Throughout this proof let
\begin{equation}
  \llabel{defM}
  M = \sup\{\abs{a_n}\colon n\in\N\}\dott
\end{equation}
```

Proof of Theorem 3. Throughout this proof let

$$M = \sup\{|a_n| \colon n \in \mathbb{N}\}. \tag{8}$$

```
\argument{
  the assumption that $a\in\BoundedSeqs$
} {
  that $M<\infty$\dott
}
```

Observe that the assumption that $a \in \ell^\infty$ establishes that $M < \infty$.

```
\argument{
  \lref{defM}
} {
  that for all
    $n\in\N$
  it holds that
  \begin{equation}
    \llabel{eq1}
    a_n\in[-M,M]
    \dott
  \end{equation}
}
```

Moreover, note that (8) ensures that for all $n \in \mathbb{N}$ it holds that

$$a_n \in [-M, M]. \tag{9}$$

```
\argument{
  \lref{eq1}
} {
  that
  \begin{equation}
    \llabel{ind1}
    \abs{\{n\in\N\colon a_n\in [-M,M]\}}=\infty\dott
  \end{equation}
}
```

Hence, we obtain that

$$|\{n \in \mathbb{N}\colon a_n \in [-M, M]\}| = \infty. \tag{10}$$

```
\argument{
  the fact that for all
    $c\in\R$,
    $d\in\R\cap(c,\infty)$
  it holds that
  \begin{equation}
    \{n\in\N\colon a_n\in[c,d]\}
    \subseteq
    \cu*{n\in\N\colon a_n\in\br*{c,\tfrac{c+d}2}}
    \cup
    \cu*{n\in\N\colon a_n\in\br*{\tfrac{c+d}2,d}}
  \end{equation}
} {
  that for all
    $c\in\R$,
    $d\in\R\cap[c,\infty)$
    such that
      $\abs{\{n\in\N\colon a_n\in[c,d]\}}=\infty$
  it holds that
  \begin{equation}
    \llabel{ind2}
    \abs*{\cu*{n\in\N\colon a_n\in\br*{c,\tfrac{c+d}2}}}=\infty
    \;\lor\;
    \abs*{\cu*{n\in\N\colon a_n\in\br*{\tfrac{c+d}2,d}}}=\infty
    \dott
  \end{equation}
}
```

Furthermore, observe that the fact that for all $c \in \mathbb{R}$, $d \in \mathbb{R} \cap (c, \infty)$ it holds that

$$\{n \in \mathbb{N}\colon a_n \in [c, d]\} \subseteq \left\{n \in \mathbb{N}\colon a_n \in \left[c, \tfrac{c+d}{2}\right]\right\} \cup \left\{n \in \mathbb{N}\colon a_n \in \left[\tfrac{c+d}{2}, d\right]\right\} \tag{11}$$

implies that for all $c \in \mathbb{R}$, $d \in \mathbb{R} \cap [c, \infty)$ such that $|\{n \in \mathbb{N} \colon a_n \in [c,d]\}| = \infty$ it holds that

$$\left|\left\{n \in \mathbb{N} \colon a_n \in \left[c, \tfrac{c+d}{2}\right]\right\}\right| = \infty \ \vee \ \left|\left\{n \in \mathbb{N} \colon a_n \in \left[\tfrac{c+d}{2}, d\right]\right\}\right| = \infty. \tag{12}$$

```
\argument{
  \lref{ind1};
  \lref{ind2}
} {
  that there exist
    $c=(c_n)_{n\in\N}\colon \N\to\R$
    and
    $d=(d_n)_{n\in\N}\colon \N\to\R$
  which satisfy for all
    $n\in\N$
  that
  \begin{gather}
    \llabel{defcd1}
    (c_1,d_1)
    =
    (-M,M)
    ,\quad
    (c_{n+1},d_{n+1})
    \in\{(c_n, \tfrac{c_n+d_n}2),(\tfrac{c_n+d_n}2,d_n)\}
    ,\\
    \llabel{defcd2}
    \text{and}\quad
    \abs*{\cu*{k\in\N\colon a_k\in\br*{c_n,d_n}}}
    =
    \infty
    \dott
  \end{gather}
}
```

Combining this and (10) shows that there exist $c = (c_n)_{n \in \mathbb{N}} \colon \mathbb{N} \to \mathbb{R}$ and $d = (d_n)_{n \in \mathbb{N}} \colon \mathbb{N} \to \mathbb{R}$ which satisfy for all $n \in \mathbb{N}$ that

$$(c_1, d_1) = (-M, M), \quad (c_{n+1}, d_{n+1}) \in \{(c_n, \tfrac{c_n+d_n}{2}), (\tfrac{c_n+d_n}{2}, d_n)\}, \tag{13}$$

$$\text{and} \quad |\{k \in \mathbb{N} \colon a_k \in [c_n, d_n]\}| = \infty. \tag{14}$$

```
\startnewargseq
\argument{
  \lref{defcd1};
  induction
} {
  that for all
    $n\in\N$
  it holds that
    \llabel{cdbounded}
    $c_n,d_n\in [-M,M]$\dott
}
```

Note that (13) and induction prove that for all $n \in \mathbb{N}$ it holds that $c_n, d_n \in [-M, M]$.

```
\argument{
  \lref{cdbounded}
} {
  that
  \begin{equation}
    \llabel{cbounded}
    c\in\BoundedSeqs\dott
  \end{equation}
}
```

Hence, we obtain that

$$c \in \ell^\infty. \tag{15}$$

```
\argument{
  \lref{defcd1}
} {
  that for all
    $n\in\N$
  it holds that
    \llabel{cincreasing}
    $c_{n+1}\geq c_n$\dott
}
```

Moreover, observe that (13) demonstrates that for all $n \in \mathbb{N}$ it holds that $c_{n+1} \geq c_n$.

```
\argument{
  \lref{cbounded};
  \lref{cincreasing};
  \cref{lem:monoconv};
} {
  that $c$ converges and
  \begin{equation}
    \llabel{limc}
    \lim_{n\to\infty} c_n
    =
    \sup(\{c_n\colon n\in\N\})
    \dott
  \end{equation}
}
```

Combining this, (15), and Lemma 2 establishes that c converges and

$$\lim_{n\to\infty} c_n = \sup(\{c_n \colon n \in \mathbb{N}\}). \tag{16}$$

```
\argument{
  \lref{defcd2};
  induction;
} {
  that there exists
    $m=(m_k)_{k\in\N}\colon\N\to\N$
  which satisfies that for all
    $k\in\N$
  it holds that
    $m_{k+1}>m_k$
  and
  \begin{equation}
    \llabel{defm}
    a_{m_k}\in[c_k,d_k]\dott
  \end{equation}
}
```

In the next step we combine (14) and induction to obtain that there exists $m = (m_k)_{k\in\mathbb{N}} \colon \mathbb{N} \to \mathbb{N}$ which satisfies that for all $k \in \mathbb{N}$ it holds that $m_{k+1} > m_k$ and

$$a_{m_k} \in [c_k, d_k]. \tag{17}$$

```
\argument{
  \lref{defcd1}
} {
  that for all
    $n\in\N$
  it holds that
    \llabel{distcd}
    $d_{n+1}-c_{n+1}=\tfrac{d_n-c_n}2$\dott
}
```

In the next step we note that (13) ensures that for all $n \in \mathbb{N}$ it holds that $d_{n+1} - c_{n+1} = \frac{d_n - c_n}{2}$.

```
\argument{
  \lref{distcd};
  the fact that $d_1-c_1=2M$;
  induction;
} {
  that for all
    $n\in\N$
  it holds that
    \llabel{distcd2}
    $d_n-c_n=2^{2-n}M$\dott
}
```

This, the fact that $d_1 - c_1 = 2M$, and induction imply that for all $n \in \mathbb{N}$ it holds that $d_n - c_n = 2^{2-n}M$.

```
\argument{
  \lref{distcd2};
  \lref{defm};
} {
  that for all
    $k\in\N$
  it holds that
  \begin{equation}
    \llabel{amkabove}
    a_{m_k}
    \leq
    d_k
    \leq
    c_k + 2^{2-k}M
    \leq
    \sup(\{c_n\colon n\in\N\}) + 2^{2-k}M
    \dott
  \end{equation}
}
```

This and (17) show that for all $k \in \mathbb{N}$ it holds that

$$a_{m_k} \leq d_k \leq c_k + 2^{2-k}M \leq \sup(\{c_n \colon n \in \mathbb{N}\}) + 2^{2-k}M. \tag{18}$$

```
\argument{
  \lref{amkabove}
} {
  that
  \begin{equation}
    \llabel{limsup}
    \limsup_{k\to\infty} a_{m_k}
    \leq
    \sup(\{c_n\colon n\in\N\})
    \dott
  \end{equation}
}
```

Hence, we obtain that

$$\limsup_{k\to\infty} a_{m_k} \leq \sup(\{c_n \colon n \in \mathbb{N}\}). \tag{19}$$

```
\argument{
  \lref{defm};
  \lref{limc};
} {
  that
  \begin{equation}
    \llabel{liminf}
    \liminf_{k\to\infty} a_{m_k}
    \geq
    \liminf_{k\to\infty} c_k
    =
    \sup(\{c_n\colon n\in\N\})
    \dott
  \end{equation}
}
```

In the next step we observe that (17) and (16) prove that

$$\liminf_{k\to\infty} a_{m_k} \geq \liminf_{k\to\infty} c_k = \sup(\{c_n \colon n \in \mathbb{N}\}). \tag{20}$$

```
\argument{
  \lref{limsup};
  \lref{liminf};
} {
  that $(a_{m_k})_{k\in\N}$ converges\dott
}
\end{aproof}
```

This and (19) demonstrate that $(a_{m_k})_{k\in\mathbb{N}}$ converges. The proof of Theorem 3 is thus complete. □