Definition 1. We denote by ℓ^∞ the set which satisfies

$$\ell^\infty = \{(a = (a_n)_{n\in\mathbb{N}} \colon \mathbb{N} \to \mathbb{R}) \colon \sup\{|a_n| \colon n \in \mathbb{N}\} < \infty\}. \tag{1}$$

Lemma 2. *Let $a \in \ell^\infty$ and assume that for all $n \in \mathbb{N}$ it holds that $a_{n+1} \geq a_n$ (cf. Definition 1). Then a converges and*

$$\lim_{n\to\infty} a_n = \sup(\{a_n \colon n \in \mathbb{N}\}). \tag{2}$$

Proof of Lemma 2. Throughout this proof let

$$L = \sup(\{a_n \colon n \in \mathbb{N}\}). \tag{3}$$

Observe that (3) ensures that for all $n \in \mathbb{N}$ it holds that

$$a_n \leq L. \tag{4}$$

Next, note that (3) and the assumption that $a \in \ell^\infty$ imply that $L < \infty$. Combining this and (3) shows that for all $\varepsilon \in (0,\infty)$ there exists $m \in \mathbb{N}$ such that

$$a_m > L - \varepsilon. \tag{5}$$

In the next step we combine the assumption that for all $n \in \mathbb{N}$ it holds that $a_{n+1} \geq a_n$ and induction to obtain that for all $m \in \mathbb{N}$, $n \in \mathbb{N} \cap [m,\infty)$ it holds that

$$a_n \geq a_m. \tag{6}$$

This and (5) prove that for all $\varepsilon \in (0,\infty)$ there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N} \cap [m,\infty)$ it holds that $a_n > L - \varepsilon$. This and (4) demonstrate that for all $\varepsilon \in (0,\infty)$ there exists $m \in \mathbb{N}$ such that for all $n \in \mathbb{N} \cap [m,\infty)$ it holds that

$$|a_n - L| < \varepsilon. \tag{7}$$

Hence, we obtain that a converges and $\lim_{n\to\infty} a_n = L$. The proof of Lemma 2 is thus complete. □

Theorem 3 (Bolzano–Weierstrass theorem)**.** *Let $a \in \ell^\infty$ (cf. Definition 1). Then there exists $n = (n_k)_{k\in\mathbb{N}} \colon \mathbb{N} \to \mathbb{N}$ such that it holds for all $k \in \mathbb{N}$ that $n_{k+1} > n_k$ and $(a_{n_l})_{l\in\mathbb{N}}$ converges.*

Proof of Theorem 3. Throughout this proof let

$$M = \sup\{|a_n| \colon n \in \mathbb{N}\}. \tag{8}$$

Observe that the assumption that $a \in \ell^\infty$ establishes that $M < \infty$. Moreover, note that (8) ensures that for all $n \in \mathbb{N}$ it holds that

$$a_n \in [-M, M]. \tag{9}$$

Hence, we obtain that

$$|\{n \in \mathbb{N} \colon a_n \in [-M, M]\}| = \infty. \tag{10}$$

Furthermore, observe that the fact that for all $c \in \mathbb{R}$, $d \in \mathbb{R} \cap (c, \infty)$ it holds that

$$\{n \in \mathbb{N} \colon a_n \in [c,d]\} \subseteq \left\{n \in \mathbb{N} \colon a_n \in \left[c, \tfrac{c+d}{2}\right]\right\} \cup \left\{n \in \mathbb{N} \colon a_n \in \left[\tfrac{c+d}{2}, d\right]\right\} \quad (11)$$

implies that for all $c \in \mathbb{R}$, $d \in \mathbb{R} \cap [c, \infty)$ such that $|\{n \in \mathbb{N} \colon a_n \in [c,d]\}| = \infty$ it holds that

$$\left|\left\{n \in \mathbb{N} \colon a_n \in \left[c, \tfrac{c+d}{2}\right]\right\}\right| = \infty \;\vee\; \left|\left\{n \in \mathbb{N} \colon a_n \in \left[\tfrac{c+d}{2}, d\right]\right\}\right| = \infty. \quad (12)$$

Combining this and (10) shows that there exist $c = (c_n)_{n \in \mathbb{N}} \colon \mathbb{N} \to \mathbb{R}$ and $d = (d_n)_{n \in \mathbb{N}} \colon \mathbb{N} \to \mathbb{R}$ which satisfy for all $n \in \mathbb{N}$ that

$$(c_1, d_1) = (-M, M), \quad (c_{n+1}, d_{n+1}) \in \{(c_n, \tfrac{c_n+d_n}{2}), (\tfrac{c_n+d_n}{2}, d_n)\}, \quad (13)$$

$$\text{and} \quad |\{k \in \mathbb{N} \colon a_k \in [c_n, d_n]\}| = \infty. \quad (14)$$

Note that (13) and induction prove that for all $n \in \mathbb{N}$ it holds that $c_n, d_n \in [-M, M]$. Hence, we obtain that

$$c \in \ell^\infty. \quad (15)$$

Moreover, observe that (13) demonstrates that for all $n \in \mathbb{N}$ it holds that $c_{n+1} \geq c_n$. Combining this, (15), and Lemma 2 establishes that c converges and

$$\lim_{n \to \infty} c_n = \sup(\{c_n \colon n \in \mathbb{N}\}). \quad (16)$$

In the next step we combine (14) and induction to obtain that there exists $m = (m_k)_{k \in \mathbb{N}} \colon \mathbb{N} \to \mathbb{N}$ which satisfies that for all $k \in \mathbb{N}$ it holds that $m_{k+1} > m_k$ and

$$a_{m_k} \in [c_k, d_k]. \quad (17)$$

In the next step we note that (13) ensures that for all $n \in \mathbb{N}$ it holds that $d_{n+1} - c_{n+1} = \frac{d_n - c_n}{2}$. This, the fact that $d_1 - c_1 = 2M$, and induction imply that for all $n \in \mathbb{N}$ it holds that $d_n - c_n = 2^{2-n}M$. This and (17) show that for all $k \in \mathbb{N}$ it holds that

$$a_{m_k} \leq d_k \leq c_k + 2^{2-k}M \leq \sup(\{c_n \colon n \in \mathbb{N}\}) + 2^{2-k}M. \quad (18)$$

Hence, we obtain that

$$\limsup_{k \to \infty} a_{m_k} \leq \sup(\{c_n \colon n \in \mathbb{N}\}). \quad (19)$$

In the next step we observe that (17) and (16) prove that

$$\liminf_{k \to \infty} a_{m_k} \geq \liminf_{k \to \infty} c_k = \sup(\{c_n \colon n \in \mathbb{N}\}). \quad (20)$$

This and (19) demonstrate that $(a_{m_k})_{k \in \mathbb{N}}$ converges. The proof of Theorem 3 is thus complete. □